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A Note on the Solution of Rational Expectations Models with One Future Variable

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Abstract

In this note, we propose a short-cut to the solution of linear rational expectations models with one future variable. We take a version of Cagan's (1956) hyperinflation model as a case study. Our solution makes use of the martingale property that given the information set at time t-1, the rational expectation of a variable formed under this set will be the same for time t and for time t+1. This result can also be derived from the error orthogonality property of the rational expectations models. This short-cut might also prove useful in simplifying the econometric estimation of rational expectations models similar in structure to Cagan's (1956) hyperinflation model.

Keywords: Rational expectations models, Martingale property, Cagan's hyperinflation model
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1. Introduction

In this paper, we provide a short-cut for solving linear rational expectations models with one future variable (REFV) by making use of martingales and martingale difference properties. The proposed method is illustrated with the aid of the Minford's (1992) version of Cagan's (1956) hyperinflation model, described by the equation system (1) – (3).

$$m_t = p_t + y_t - \alpha\,[E_{t-1}p_{t+1} - E_{t-1}p_t],\ (\alpha > 0) \quad (1)$$

$$p_t = E_{t-1}p_t + \delta\,(y_t - y^*) \quad (2)$$

$$m_t = m^* + \varepsilon_t \quad (3)$$

where m_t, p_t, and y_t are the natural logarithms of money supply, price level, and output, respectively; and m^* is a monetary target, y^* is normal output, ε_t is a well-behaved error term with $\varepsilon \sim N(0, \sigma^2)$, and E is the rational expectations operator.

The first equation is a money demand function, specifying that the demand for money responds negatively to expected price level changes in a hyperinflationary environment. This specification is one of the original contributions of Cagan's (1956) study of the European hyperinflations of the 1920s. The second equation is a Phillips-curve type relationship, and the last equation is a money supply function, where the government has a monetary target, m*, with a well-behaved error term, ε.

The above model is an example of RE models involving a future variable, and the main problem in solving the model comes from the presence of $E_{t-1}p_{t+1}$ in the first equation. Various approaches have been offered in the literature to tackle this problem. Minford (1992, Chapter 2) and Broze and Szafarz (1991, Chapter, 2.2) provide a comprehensive survey of these approaches; namely, a basic method, Sargent and Wallace's (1975) method of forward substitution, Muth's (1961) method of undetermined coefficients, Lucas's (1972) method of undetermined coefficients, Blanchard and Kahn's (1980) perfect foresight solution, and the autoregressive-moving average solutions. Nevertheless, the solution methods offered by the above authors, even for the simple the model described in equations (1)-(3), are generally quite involved.

2. A Simplified Solution

Our simplification of the solution to the system of Equations (1)-(3) runs on the following grounds. First, we make use of the fact that the stochastic process p_t is a martingale, i.e.,

$$E[p_t | I_t] = p_t, \text{ and} \quad (4)$$

$$E[p_{t+j} | I_t] = p_t, \forall j > 0 \quad (5)$$

where I_t is the information set available at time t.[1]

Next, we re-write equation (1) as:

$$m_t = p_t + y_t - \alpha [E_{t-1}(p_{t+1} - p_t)], (\alpha > 0) \quad (6)$$

or,

$$m_t = p_t + y_t - \alpha[E_{t-1}\varepsilon_{t+1}], (\alpha > 0), \quad (7)$$

where the error orthogonality property in the RE models implies that $E_{t-1}\varepsilon_{t+1} = 0$.[2]

Based on equations (4) and (5) or (6) and (7), one can then temporarily eliminate the $[E_{t-1}p_{t+1} - E_{t-1}p_t]$ term from equation (1). Hence, equation (1) reduces to a simpler one, i.e., $m_t = p_t + y_t$, and the solution for the system becomes almost immediate. As a result, the following solutions, demonstrating the neutrality of anticipated monetary policy, are obtained.

$$E_{t-1}p_t = E_{t-1}p_{t+1} = m^* - y^* \quad (8)$$

$$y_t = y^* + (1/1+\delta)\, \varepsilon_t \quad (9)$$

$$p_t = m^* - y^* + (\delta/1+\delta)\, \varepsilon_t \quad (10)$$

Note that the same solutions given in equations (8)-(10) are obtained regardless of whether the term $(E_{t-1}p_{t+1} - E_{t-1}p_t)$ appears in equation (1) or not.[3]

3. Implications of the Simplified Solution and Concluding Remarks

The suggested simplified solution to the model described above by making use of martingales and martingale difference properties to set $(E_{t-1}p_{t+1} - E_{t-1}p_t)$ in equation (1) or $E_{t-1}\varepsilon_{t+1}$ in equation (7) to zero is indeed equal to imposing a stability condition on the model to ensure a unique saddlepath equilibrium.[4] Minford (1992, pp. 20-23), for example, demonstrates that setting $Ep_{t+i+1} - Ep_{t+i} = 0$ ($i \geq N$, where N is the time period after which the inflation expectations are zero) is necessary for obtaining a unique stable solution for the model. Sargent and Wallace (1975, eqs. 24 and 25) also argue for a similar result. Our approach, however, imposes the stability and uniqueness conditions for this simple linear RE model with one future variable by explicitly making use of the martingale property that given the information set at time t-1, the rational expectation of a variable formed under this information set

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[1] See, for example, Broze and Szafarz (1991, pp. 8-11) and the references cited therein.

[2] In the martingale terminology, the error term ε_t follows a martingale difference process if $E[\varepsilon_t | I_t] = \varepsilon_t$, and $E[\varepsilon_t+j \; | I_t] = 0$, $j > 0$, where I_t is the information set available at time t.

[3] See Minford (1992, pp. 15-24).

[4] See Driskill (2006) for a review of the problem of multiple equilibria in rational expectations models and the problems associated with imposing stability and uniqueness conditions.

will be the same for time t and time t+1. Or, in general, $E_{t-1}p_{t+N+1} = E_{t-1}p_{t+N}$ under stability and uniqueness conditions.

Furthermore, our method has implications for applied researchers since the econometric estimation of REFV models is normally quite cumbersome. Cagan's (1956) model and its variants, for example, have been popularly used in empirical work. In an attempt to simplify the estimation issues, Taylor (1991) imposed the restriction that the expected inflation in period t+1 is the same as the one in period t. Taylor's (1991) justifies this by the property that the forecast error should be stationary in the statistical sense. Taylor (1991) further demonstrates that the stationarity of forecasts errors introduces a long-run (cointegration) restriction into Cagan's model. Nevertheless, examining Taylor's (1991) implementation of Cagan's (1956) model in a textbook example context, Stewart (2005, pp. 816-817) shows that the money supply and price level series for Germany (September 1920 – July 1923) and Poland (April 1922 – November 1923) are not cointegrated. Thus, Taylor's cointegrating restriction assumption might be questionable at least on empirical grounds. Perhaps, the time span of the data is too short to yield meaningful inferences about long-run relationships.

Compared to Taylor's approach, we do not place our restrictions from an empirical aspect, but rather make explicit use of the error orthogonality property in RE models, i.e., $E_{t-1}\varepsilon_{t+1} = 0$, to eliminate the $E_{t-1}p_{t+1} - E_{t-1}p_t$ term from equation (1) temporarily. This facilitates the mathematical solution of the model and leads to an estimable econometric specification.

It should, however, be stated that our suggested simplification is valid for rational expectations models with only one future variable. In general, solving a RE model entails the specification of time paths for the endogenous variables. Thus, choosing a number of periods, N, that tends to infinity is necessary in order to check the stability of the model solution. In RE models with more than one future variable, the martingale property does not necessarily lead to the conclusion that the model's stability and uniqueness conditions are fulfilled at the same time. Nevertheless, our short-cut still works as an operational tool for solving and estimating RE models with one future variable, which are commonly used in the economics and finance literature.

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